Linear systems – Final exam (solutions)

Final exam 2018–2019, Tuesday 18 June 2019, 9:00 – 12:00

Problem 1 (10 points)

To solve the initial value problem

$$t\dot{x}(t) - x(t) = t^3 \sin(t^2), \qquad x(\sqrt{\pi}) = 0, \tag{1}$$

consider the differential equation for $t > 0$ and write it in the standard form

$$\dot{x}(t) = \frac{1}{t}x(t) + t^2 \sin(t^2), \tag{2}$$

which is achieved after dividing by t.

Computation of the integrating factor yields

$$F(t) = \int \frac{1}{t}\,\mathrm{d}t = \ln|t|, \tag{3}$$

such that

$$e^{-F(t)} = e^{-\ln|t|} = e^{\ln\frac{1}{|t|}} = \frac{1}{|t|} = \frac{1}{t}, \tag{4}$$

where again $t > 0$ is used. It now follows that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\big\{t^{-1}x(t)\big\} = -t^{-2}x(t) + t^{-1}\dot{x}(t) = \frac{t\dot{x}(t) - x(t)}{t^2}, \tag{5}$$

such that substitution of the differential equation gives

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\big\{t^{-1}x(t)\big\} = \frac{t^3\sin(t^2)}{t^2} = t\sin(t^2). \tag{6}$$

Hence,

$$\frac{x(t)}{t} = \int t\sin(t^2)\,\mathrm{d}t = \int \tfrac{1}{2}\sin(u)\,\mathrm{d}u = -\tfrac{1}{2}\cos(u) + C = -\tfrac{1}{2}\cos(t^2) + C, \tag{7}$$

where we have used the substitution $u = t^2$. Consequently, the general solution to the differential equation is given by

$$x(t) = -\tfrac{1}{2}t\cos(t^2) + Ct, \tag{8}$$

which can be verified to satisfy the differential equation in (1) for all $t \in \mathbb{R}$.

To solve the initial value problem, evaluate

$$x(\sqrt{\pi}) = -\tfrac{1}{2}\sqrt{\pi}\cos(\pi) + C\sqrt{\pi} = (\tfrac{1}{2} + C)\sqrt{\pi}. \tag{9}$$

As $x(\sqrt{\pi}) = 0$, this leads to

$$C = -\tfrac{1}{2}. \tag{10}$$

Problem 2 (3 + 3 + 6 = 12 points)

Consider the scalar differential equation

$$\frac{\mathrm{d}^3 q}{\mathrm{d}t^3}(t) + q^2(t)\frac{\mathrm{d}^2 q}{\mathrm{d}t^2}(t) + q(t) - q^2(t) = u(t). \tag{11}$$

(a) To write the system in state-space form, introduce the state

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \\ \ddot{q} \end{bmatrix}. \tag{12}$$

The corresponding dynamics is then given by

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \\ \dddot{q} \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ -q^2\ddot{q} - q + q^2 + u \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ -x_1^2 x_3 - x_1 + x_1^2 + u \end{bmatrix}, \tag{13}$$

which we will denote as

$$\dot{x}(t) = f(x(t), u(t)) \tag{14}$$

with

$$f(x,u) = \begin{bmatrix} x_2 \\ x_3 \\ -x_1^2 x_3 - x_1 + x_1^2 + u \end{bmatrix}. \tag{15}$$

(b) After denoting $\bar{u} = 0$ and

$$\bar{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{16}$$

it is easily verified that

$$f(\bar{x}, \bar{u}) = 0, \tag{17}$$

i.e., (16) is an equilibrium point corresponding to $\bar{u} = 0$.

To find the other equilibria, solve $0 = f(x, \bar{u})$ as

$$0 = \begin{bmatrix} x_2 \\ x_3 \\ -x_1^2 x_3 - x_1 + x_1^2 + \bar{u} \end{bmatrix}. \tag{18}$$

The first two components immediately give $x_2 = 0$ and $x_3 = 0$, after which the third component (with $\bar{u} = 0$) reads

$$0 = -x_1 + x_1^2 = x_1(x_1 - 1). \tag{19}$$

This has the two solutions $x_1 = 1$ and $x_1 = 0$, of which the first leads to the equilibrium (16). The second gives a new equilibrium, namely

$$\bar{x} = 0. \tag{20}$$

(c) To determine the linearization of (14) around the equilibrium (16), define the perturbations from the equilibrium as

$$\tilde{x} = x - \bar{x}, \qquad \tilde{u} = u - \bar{u}, \tag{21}$$

such that we have

$$\dot{\tilde{x}} = f(\bar{x} + \tilde{x}, \bar{u} + \tilde{u}). \tag{22}$$

This leads to the linearized dynamics (by the Taylor expansion)

$$\dot{\tilde{x}} = \frac{\partial f}{\partial x}(\bar{x}, \bar{u})\tilde{x} + \frac{\partial f}{\partial u}(\bar{x}, \bar{u})\tilde{u}. \tag{23}$$

Then, computation of the Jacobian of f with respect to x gives

$$\frac{\partial f}{\partial x}(x, u) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2x_1x_3 - 1 + 2x_1 & 0 & -x_1^2 \end{bmatrix}, \tag{24}$$

leading to

$$\frac{\partial f}{\partial x}(\bar{x}, \bar{u}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}. \tag{25}$$

Similarly, we obtain

$$\frac{\partial f}{\partial u}(\bar{x}, \bar{u}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{26}$$

Finally, the substitution of the results (25) and (26) in (23) gives

$$\dot{\tilde{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tilde{u}(t). \tag{27}$$

Problem 3 (16 points)

Consider the linear system given by the transfer function

$$T(s) = \frac{s}{s^4 + as^3 + as^2 + as + 1}, \tag{28}$$

where $a \in \mathbb{R}$.

To determine the values of a for which the system is externally stable, recall that external stability is characterized by the poles of the transfer function (28). These are the roots of the denominator polynomial *after common factors have been canceled.*

As a first step, note that

$$p(s) = s, \qquad q(s) = s^4 + as^3 + as^2 + as + 1, \tag{29}$$

have no common factors. Namely, the only root of p is at 0 and we have that $q(0) = 1$ regardless of the value of a, i.e., p and q are coprime.

Thus, the linear system given by (28) is externally stable if and only if the polynomial q is stable. Stability of q can be evaluated through the following Routh table.

	s^4	s^3	s^2	s^1	s^0	
$a\times$	1	a	a	a	1	q
$1\times$	a		a			
		a^2	$a(a-1)$	a^2	a	result of step 1
$(a-1)\times$		a	$a-1$	a	1	after division by a: q'
$a\times$		$a-1$		1		
			$(a-1)^2$	$a(a-2)$	$a-1$	result of step 2: q''

Note that a necessary condition for stability of q is that all coefficients have the same sign. This immediately implies that

$$a > 0 \tag{30}$$

is necessary for stability.

This condition also implies that the result of step 1 can be divided by a. Then, following the same reasoning, a necessary condition of the result of step 1 to be stable is that

$$a > 1. \tag{31}$$

Similarly, after step 2, we obtain the necessary condition

$$a > 2. \tag{32}$$

However, the result of step 2 is a quadratic polynomial which is known to be stable if and only if all coefficients have the same sign. This means that (32) is also a sufficient condition for stability of the polynomial q''.

Consequently, by the Routh-Hurwitz criterion, a necessary and sufficient condition for external stability is that

$$a > 2. \tag{33}$$

Problem 4 (4 + 12 + 8 = 24 points)

Consider the linear system

$$\mathbf{\Sigma}: \quad \dot{x}(t) = Ax(t) + Bu(t),$$

with state $x(t) \in \mathbb{R}^3$, input $u(t) \in \mathbb{R}$, and

$$A = \begin{bmatrix} -1 & -3 & -3 \\ 1 & 1 & 1 \\ -2 & -5 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

(a) To determine controllability, compute

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ -1 & -2 & -4 \end{bmatrix}. \tag{34}$$

It is clear that

$$\operatorname{rank} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ -1 & -2 & -4 \end{bmatrix} = 1 < 3 = n, \tag{35}$$

such that the system is not controllable.

(b) To find the desired transformation, first note that the reachable subspace is given as

$$\mathcal{W} = \operatorname{im} \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \operatorname{im} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ -1 & -2 & -4 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}, \tag{36}$$

such that

$$q_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \tag{37}$$

forms a basis for the reachable subspace $\mathcal{W}$. This basis can be extended to a basis of $\mathbb{R}^3$ by choosing, for example,

$$q_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad q_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{38}$$

such that $\operatorname{span}\{q_1, q_2, q_3\} = \mathbb{R}^3$. After forming

$$T^{-1} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \tag{39}$$

solving the linear system

$$AT^{-1} = T^{-1} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \tag{40}$$

for A_{11}, A_{12}, and A_{22} leads to

$$\begin{bmatrix} -1 & -3 & -3 \\ 1 & 1 & 1 \\ -2 & -5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & -3 \\ 0 & 1 & 1 \\ 2 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & -3 \\ 0 & 1 & 1 \\ 0 & -8 & -3 \end{bmatrix}. \tag{41}$$

Thus, we have that

$$A_{11} = 2, \qquad A_{12} = \begin{bmatrix} -3 & -3 \end{bmatrix}, \qquad A_{22} = \begin{bmatrix} 1 & 1 \\ -8 & -3 \end{bmatrix}. \tag{42}$$

Similarly, solving

$$B = T^{-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \tag{43}$$

immediately leads to

$$B_1 = 1. \tag{44}$$

The pair (A_{11}, B_1) is controllable by construction.

Finally, the eigenvalues of A_{22} are obtained through

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 \\ 8 & \lambda + 3 \end{vmatrix} = (\lambda - 1)(\lambda + 3) + 8 = \lambda^2 + 2\lambda + 5, \tag{45}$$

which gives

$$\lambda = -1 \pm \tfrac{1}{2}\sqrt{4 - 4 \cdot 5} = -1 \pm \tfrac{1}{2}\sqrt{-16} = -1 \pm 2i. \tag{46}$$

Note that the matrices A_{12} and A_{22} are dependent on the choice of basis (38), but the eigenvalues of A_{22} are not.

(c) Note that

$$T(A + BF)T^{-1} = TAT^{-1} + TBFT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} FT^{-1}. \tag{47}$$

As $\sigma(A_{22}) \subset \mathbb{C}_-$, it is sufficient to choose FT^{-1} of the form

$$FT^{-1} = \begin{bmatrix} F_1 & 0 \end{bmatrix}, \tag{48}$$

in which case

$$T(A + BF)T^{-1} = \begin{bmatrix} A_{11} + B_1F_1 & A_{12} \\ 0 & A_{22} \end{bmatrix} \tag{49}$$

and

$$\sigma(A + BF) = \sigma(T(A + BF)T^{-1}) = \sigma(A_{11} + B_1F_1) \cup \sigma(A_{22}) \subset \mathbb{C}_- \tag{50}$$

if and only if $\sigma(A_{11} + B_1F_1) \subset \mathbb{C}_-$. However, the latter is easily achieved as A_{11} and B_1 are scalar. Namely,

$$A_{11} + B_1F_1 = 2 + F_1, \tag{51}$$

such that any $F_1 < -2$ is stabilizing.

To return to the original coordinates, solve the linear system

$$FT^{-1} = \begin{bmatrix} F_1 & 0 \end{bmatrix} \tag{52}$$

as

$$F \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} F_1 & 0 & 0 \end{bmatrix} \tag{53}$$

to obtain

$$F = \begin{bmatrix} F_1 & 0 & 0 \end{bmatrix}. \tag{54}$$

Problem 5 (4 + 6 + 2 = 12 points)

Consider the linear system

$$\dot{x}(t) = \begin{bmatrix} -8 & -10 & 0 \\ 5 & 7 & 0 \\ -6 & -10 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} u(t), \qquad y(t) = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} x(t).$$

(a) Compute

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 4 & 8 & 0 \end{bmatrix} \tag{55}$$

such that the unobservable subspace is given by

$$\mathcal{N} = \ker \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \ker \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 4 & 8 & 0 \end{bmatrix} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \tag{56}$$

As $\dim \mathcal{N} = 2 > 0$, the system is not observable.

(b) To evaluate detectability, we first need to compute the eigenvalues of the system matrix. To this end, note that the block lower-triangular form implies that

$$\sigma\left(\begin{bmatrix} -8 & -10 & 0 \\ 5 & 7 & 0 \\ -6 & -10 & -2 \end{bmatrix} \right) = \sigma\left(\begin{bmatrix} -8 & -10 \\ 5 & 7 \end{bmatrix} \right) \cup \{-2\}. \tag{57}$$

The eigenvalues of the upper-left submatrix are computed as

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda + 8 & 10 \\ -5 & \lambda - 7 \end{vmatrix} = (\lambda + 8)(\lambda - 7) + 50 = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3), \tag{58}$$

such that

$$\sigma\left(\begin{bmatrix} -8 & -10 \\ 5 & 7 \end{bmatrix} \right) = \{2, -3\}. \tag{59}$$

Thus, $\lambda = 2$ is the only unstable eigenvalue. Then, by the Hautus test,

$$\operatorname{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -10 & -10 & 0 \\ 5 & 5 & 0 \\ -6 & -10 & -4 \\ 1 & 2 & 0 \end{bmatrix} = 3, \tag{60}$$

such that the system is detectable.

(c) Yes, since the system is detectable.

Problem 6 (16 points)

Show that, for any matrix G, the matrix pair $(A - GC, C)$ is observable if and only if the matrix pair (A, C) is observable.

only if (By contraposition). Let (A, C) be not observable. By the Hautus test, this means that

$$\operatorname{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} < n, \tag{61}$$

for some eigenvalue $\lambda \in \sigma(A)$. As a result, there exists a nonzero vector v such that

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} v = 0. \tag{62}$$

Stated differently, we have that $Av = \lambda v$ and $Cv = 0$. This then implies that

$$\begin{bmatrix} A - GC - \lambda I \\ C \end{bmatrix} v = 0, \tag{63}$$

i.e., the matrix pair $(A - GC, C)$ is not observable.

if. The converse follows similarly.